

CS205b/CME306

Lecture 16

1 Incompressible Flow

1.1 Laplace Equation in 1D

Supplementary Reading: Osher and Fedkiw, §18.1, §18.2

Recall that the system of equations we must solve for incompressible flow is

$$\begin{aligned}\nabla \cdot \mathbf{u} &= 0 \\ \rho_t + \mathbf{u} \cdot \nabla \rho &= 0 \\ \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{\nabla p}{\rho} &= \mathbf{g}.\end{aligned}$$

The Laplace equation in 1D is given by

$$p_{xx} = 0.$$

The solution is simply a line

$$p = ax + b. \tag{1}$$

The values of the constant a and b are determined by boundary conditions. Assume that the domain is the interval $[0, 1]$. We may have *Dirichlet* boundary conditions, where the value of the function p is given at the boundary. For example,

$$p(0) = p_0 \quad p(1) = p_1.$$

Plugging the boundary conditions in the equation(1), we get

$$\begin{aligned}p(0) &= b = p_0 \\ p(1) &= a + b = p_1 \Rightarrow a = p_1 - p_0\end{aligned}$$

so the coefficients a and b are uniquely determined. Alternatively, *Neumann* boundary conditions specify the value of p_x at the boundary. For example,

$$p_x(0) = 0 \Rightarrow a = 0.$$

This gives us a family of lines with slope 0. To find b , we would need another piece of information. A Dirichlet boundary condition would pick out one of the lines with slope 0, thus determining

the solution. But observe that specifying two Neumann conditions could lead to no solution. For example,

$$p_x(0) = 0 \quad p_x(1) = 1.$$

These two boundary conditions are inconsistent, hence there is no solution. Another example is

$$p_x(0) = 0 \quad p_x(1) = 0.$$

In this case, the given boundary conditions are consistent, but incomplete. We still do not have enough information to identify a unique solution. The above examples illustrate the fact that in 1D, for the Laplace equation, we can determine the solution if we have two Dirichlet boundary conditions or one Neumann and one Dirichlet boundary condition, but will have either no solution or an underdetermined solution in the case of two Neumann boundary conditions.

1.2 Discretizing Laplacian of Pressure

We need to numerically solving Poisson's equation

$$p_{xx} = f(x).$$

We will also need the gradient to apply the pressure. We use second order central differencing for both. At each cell face, we approximate the pressure gradient with

$$(p_x)_{i+1/2} = \frac{p_{i+1} - p_i}{\Delta x} + O(\Delta x^2).$$

From this we use central differencing again to express the Laplacian at each grid node

$$\begin{aligned} (p_{xx})_i &= \frac{(p_x)_{i+1/2} - (p_x)_{i-1/2}}{\Delta x} + O(\Delta x^2) \\ &= \frac{p_{i+1} - 2p_i + p_{i-1}}{\Delta x^2} + O(\Delta x^2) \\ &= f_i. \end{aligned}$$

The result is a coupled linear system that we need to solve in order to determine p on the entire domain. However, we cannot write this equation as is for the grid points near the boundary since it will involve points outside of the domain. For example, assume that our domain is the interval $[0, 1]$ and that we have grid points $0, 1, \dots, M, M + 1$ uniformly spaced on the domain. The equation for p_1 is

$$\frac{p_2 - 2p_1 + p_0}{\Delta x^2} = f_1.$$

If we have a Dirichlet boundary condition specified on the left of the domain

$$p_0 = \beta,$$

then the equation for p_1 becomes

$$\frac{p_2 - 2p_1}{\Delta x^2} = f_1 - \frac{\beta}{\Delta x^2}.$$

If we have a Neumann boundary condition specified at the half grid point $\frac{1}{2}$

$$(p_x)_{\frac{1}{2}} = \alpha,$$

from which we apply central differencing again to obtain the Laplacian

$$\begin{aligned}
(\Delta p)_{i,j} &= \frac{(p_x)_{i+1/2,j} - (p_x)_{i-1/2,j}}{\Delta x} + \frac{(p_y)_{i,j+1/2} - (p_y)_{i,j-1/2}}{\Delta y} + O(\Delta x^2) + O(\Delta y^2) \\
&= \frac{p_{i+1,j} - 2p_{i,j} + p_{i-1,j}}{\Delta x^2} + \frac{p_{i,j+1} - 2p_{i,j} + p_{i,j-1}}{\Delta y^2} + O(\Delta x^2) + O(\Delta y^2) \\
&= f_{i,j}.
\end{aligned}$$

In $2D$ we need boundary conditions specified around the entire domain. If at least one boundary condition is Dirichlet, then the resulting matrix will be a banded symmetric positive definite matrix. We can use an iterative solver such as preconditioned conjugate gradients. If all the boundary conditions are Neumann, then the matrix will have a null space, and we must ensure we have a compatible system.

If the density is spatially varying, then we must use a discretization with variable coefficients. To simplify the notation slightly, let $\beta = \frac{1}{\rho}$. Then, we can write

$$\begin{aligned}
\left(\nabla \cdot \frac{1}{\rho} \nabla p \right)_{i,j} &= (\nabla \cdot \beta \nabla p)_{i,j} \\
&= \frac{\beta_{i+1/2,j} (p_x)_{i+1/2,j} - \beta_{i-1/2,j} (p_x)_{i-1/2,j}}{\Delta x} + O(\Delta x^2) \\
&\quad + \frac{\beta_{i,j+1/2} (p_y)_{i,j+1/2} - \beta_{i,j-1/2} (p_y)_{i,j-1/2}}{\Delta y} + O(\Delta y^2) \\
&= f_{i,j}.
\end{aligned}$$

Note that this will require densities at the cell walls.

1.3 Compatibility Condition

Poisson's equation with all Neumann boundary conditions must satisfy a *compatibility condition* for a solution to exist. The problem is given by

$$\begin{cases} \Delta p = f & \text{in } \Omega \\ \nabla p \cdot \mathbf{n} = g & \text{on } \partial\Omega \end{cases}$$

where \mathbf{n} is the unit normal to the boundary. From the equation we have the relations

$$\int_{\Omega} f dV = \int_{\Omega} \Delta p dV = \int_{\Omega} \nabla \cdot \nabla p dV = \int_{\partial\Omega} \nabla p \cdot \mathbf{n} dS = \int_{\partial\Omega} g dS$$

where the third equality follows from the divergence theorem. The compatibility condition is

$$\int_{\Omega} f dV = \int_{\partial\Omega} g dS.$$

The right hand side f will be of the form $f = \nabla \cdot \mathbf{u}^*$, and $g = 0$. Therefore, the compatibility condition is

$$\int_{\Omega} \nabla \cdot \mathbf{u}^* dV = \int_{\partial\Omega} \mathbf{u}^* \cdot \mathbf{n} dS = 0$$

where the first equality follows from the divergence theorem. This condition needs to be satisfied when specifying the boundary condition on \mathbf{u}^* in order to guarantee the existence of a solution.